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On stability of discrete-time predator-prey systems subject to Allee effects

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Abstract. *The stability of predator-prey system subject to the Allee effect is of considerable interest in recent times. The present investigation is an attempt to observe the stability nature of a discrete-time predator-prey system with Allee effect on predator, prey and on both the population. We have obtained the conditions for which the equilibrium points of each of the three cases are asymptotically stable. By mathematical analysis, phase plane and bifurcation analysis it is observed that, predation driven Allee on prey population has destabilizing effect on the model system and a wide range of dynamical behavior is observed from stable equilibrium points to unstable one through a regime of neutrally stable limit cycles. On the contrary, when Allee effect is present in predator only or both, the stability of the model system increases.*

Key words: Allee effect, Predator-prey system, Jurry conditions, Stability analysis, Bifurcation.

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1 Introduction

A positive relationship between population size/density and its per capita growth rate (pgr) is known as Allee effect (1). It generally happens when certain component of individual fitness (e.g. litter size, juvenile survival, adult mortality etc.) is reduced when population size decreases, known as component Allee effect (2). When two or more component Allee act together and result in lowering of the overall mean individual fitness, known as demographic Allee effect (2; 3). Component Allee may not result in a decline of population pgr as it may be balanced by negative effect of some other component of individual fitness. However, demographic Allee assumes the positive density dependence, which increases species' likelihood of extinction (4). Sometime demographic Allee may become so severe that below a threshold population size pgr becomes negative and extinction becomes almost a certain event. This is known as strong Allee effect and the threshold population size is known as Allee threshold.

Predation is a general term in ecological theory and almost universal in different ecological processes. It is also recognized that predators having a type-I or type-II functional response, may drive the prey to have Allee mechanism, without a type-III aggregative response (5). Since last few decades Allee mechanism has received much significance in general ecological theory and conservation biology. But, from both theoretical and empirical point of view it received much significance in single population dynamics compared to the interactive scenario (6). Allee effect increases the likelihood of species extinction which has become an important component in conservation biology (7; 8; 9). With alarming theoretical consequences and empirical evidences of Allee effect, it should not be ignored, perhaps should be given utmost priority in prey-predator system. Excellent reviews on Allee effect and its ecological implications can be found in (3; 10; 11). Empirical evidence of Allee effect has been reported in many natural populations including plants (12; 13), insects (14), marine invertebrates (15), birds and mammals (16).

The stability analysis of prey-predator system has great importance in understanding its biological relevance and may have strong implications in conservation management (17). In this paper, we particularly concentrate on a discrete-time prey-predator model, assuming populations have non-overlapping generations. (6) have considered the prey-predator system when prey population is subject to Allee effect and observed the stabilizing effect due to incorporation of Allee mechanism. However, predators may also experience an Allee effect as many predators may be regarded as K-strategists; in fact it may be easier for them to be susceptible to Allee effect (17; 18) and it could be strong Allee also (19). (20) considered demographic Allee in predator and using bifurcation analysis they have shown the stabilizing effect of Allee effect on model systems. We elaborate the analysis when only predator and both prey and predator are subjected to Allee effect. We have considered both Allee effect I and II, as explained by (17). Allee effect I increase the intrinsic death rate or decrease the intrinsic growth rate, where as Allee effect II is caused by anti-predator defence, for example, anti-predator vigilance and aggression (7; 21; 22).

This paper is organized as follows: In section 2.1, we investigate local behavior of the equilibrium points when predator population is subject to an Allee effect and present numerical simulations supporting the theoretical stability results. In section 2.2, we perform the same as section 2.1 when prey population in system 2.1 is subject to type-II Allee effect. In section 2.3, we do the analysis with Allee effect in both the population. Finally, the last section of the paper is devoted to the discussion and remarks.

2 Models and Results

We have considered the following model considered by (6), as base framework of the current work.

$$\begin{aligned} N_{t+1} &= N_t + rN_t(1-N_t) - aN_tP_t \\ P_{t+1} &= P_t + aP_t(N_t - P_t), \end{aligned} \tag{2.1}$$

where, r and a are positive constants. The logistic term $rN_t(1-N_t)$ represents the growth law of prey population in absence of predators, r being the usual growth rate. The term aN_tP_t is the predation term and a is termed as predation parameter. It is worth mentioning that, this particular discrete-time model considers linear functional response as predation term. Here (N_0^*, P_0^*) is the unique positive equilibrium point of (2.1), where

$$N_0^* = P_0^* = \frac{r}{a+r}, \tag{2.2}$$

which is asymptotically stable if

$$2 - \tfrac{4}{r} < \tfrac{ar}{a+r} < 1. \tag{2.3}$$

In particular, their analysis suggests that Allee effect act as a stabilizing factor for this system (2.1). By mathematical analysis and numerical simulations they have shown the impact of Allee effect on prey population and found that the stability of the model increases in presence of Allee effect.

In general Allee effect is incorporated into the population growth models by multiplying the probability $P(N)$ with the birth term of the corresponding non-Allee growth equation, where $P(N)$ is the probability that a female finds and mates with at least one male during the reproductive period. More biological rationale behind this is mentioned in (7) and (23). The probability $P(N)$ should satisfy the following basic criteria that:

1. No mating occurs at zero population size, $P(0) = 0$.
2. $P'(N) > 0$ i.e. if population size increases the probability that, a female will find a mate increases.
3. Mating is guaranteed when the population is large, that is $P(N) \to 1$ and $N \to +\infty$.

In general the functional forms for $P(N)$ are considered as, (1) $P(N) = 1 - e^{-\alpha N}$, $\alpha > 0$ (negative exponential); (2) $P(N) = \frac{N}{N+\mu}$, $\mu > 0$(rectangular hyperbolic) and (3) $P(N) = 1 - (1-a)^{cN}$, $c > 0, 0 < a < 1$ (power complement) (7). In this paper we consider the rectangular hyperbolic form to model the mate shortage resulting an Allee effect in the population under investigation. A detail biological rationale behind all these functional forms and their mathematical derivation can be found in (23).

2.1 Allee effect on predator only

We consider the predator prey system (2.1) as subject to an Allee effect on predator population and analyze the following system:

$$\begin{aligned} N_{t+1} &= N_t + rN_t(1-N_t) - aN_tP_t \\ P_{t+1} &= P_t + aP_tN_t\left(\tfrac{P_t}{b+P_t}\right) - aP_t^2, \end{aligned} \tag{2.4}$$

where we take $\frac{P_t}{b+P_t}$ as the Allee function and $b > 0$ as the Allee constant. The bigger b is, the stronger Allee effect of the predator and lower per capita growth rate for predator population (aPN is reduced to $aPN\left(\frac{P}{b+P}\right)$ at least at low density). It is to be noted that, (20) considered demographic Allee effect in predator population that is, per capita growth rate decreases at low density. However, in model (2.4) a component Allee is introduced by multiplying the term $\left(\frac{P_t}{b+P_t}\right)$ in birth term only.

Then we have three equilibrium points of the new system (2.4) as $(0,0)$, $(1,0)$ and (N_b^*, P_b^*), where

$$N_b^* = \tfrac{r+ab}{a+r} \text{ and } P_b^* = \tfrac{a(1-b)}{a+r}. \tag{2.5}$$

For equilibrium points N_b^* and P_b^* to be positive we have the condition

$$b < 1. \tag{2.6}$$

It is to be noted that, the steady state population size for prey is increased ($N_b^* > N^*$) due to Allee effect in predator which supports the results of (17). Due to Allee effect in predator, there may be less interaction with prey and less number of prey are caught and consumed by the predator, that may direct to an increase in prey steady state population size. For the equilibrium point (0,0), the corresponding characteristic equation is $\lambda^2 - (2+r)\lambda + (1+r) = 0$ and its roots are $\lambda_1 = 1+r$, $\lambda_2 = 1$ that means (0,0) is not asymptotically stable. Similarly, we have the characteristic equation at the point (1,0) as $\lambda^2 + (r-2)\lambda + (1-r) = 0$. The roots are $\lambda_1 = 1$ and $\lambda_2 = 1-r$, so $(1,0)$ is also not asymptotically stable equilibrium point of (2.4).

Under condition (2.6), the predator-prey system (2.4) has unique positive equilibrium point (N_b^*, P_b^*). It is clear that the points N_b^* $> N_0^*$ and $P_b^* < P_0^*$. In other words, prey population density is increased and predator population density is decreased due to Allee effect on predator. It is easy to show that the equilibrium is changed from neutral stable state to stable by Allee effect. The trajectories of the prey and predator population are shown in Figure 1. After some simple calculation, the coefficient matrix of (2.4) turns out to be

$$J_b = \begin{bmatrix} 1 - rN_b^* & -aN_b^* \\ a\frac{P_b^*}{N_b^*} & 1 + a\frac{P_b^*}{N_b^*}(2b + P_b^*) - 2aP_b^* \end{bmatrix}.$$

Then we have the corresponding characteristic equation of the matrix J_b as follows:

$$P_b(\lambda) = \lambda^2 - tr(J_b)\lambda + det(J_b), \tag{2.7}$$

where

$$tr(J_b) = 2 - rN_b^* + a\frac{P_b^*}{N_b^*}(2b + P_b^*) - 2aP_b^*,$$

and

$$det(J_b) = 1 + a\frac{P_b^*}{N_b^*}(2b + P_b^*) - 2aP_b^* - rN_b^* - arP_b^*(2b + P_b^*) + 2arP_b^*N_b^* + a^2(P_b^*)^2.$$

By using the Jurry conditions we obtain that the modulus of all roots of equation (2.4) is less than 1 (i.e., the equilibrium point (N_b^*, P_b^*) is locally asymptotically stable) if

$$P_b(1) > 0,\ P_b(-1) > 0 \text{ and } det(J_b) < 1.$$

Now we derive the conditions under which the equilibrium point (N_b^*, P_b^*) is asymptotically stable. First we observe that $P_b(1) > 0$ holds if and only if $\frac{ar^2}{a+r}(1-b) > 0$, which is true by (2.6).

We now investigate the condition $P_b(-1) > 0$ and $det(J_b) < 1$ under the condition (2.6). We see that

$$P_b(-1) > 0 \Leftrightarrow 2 - \frac{4(r+ab)}{r(r+ab^2)} < \frac{ar(1-b)^2(r+ab)}{(r+ab^2)(a+r)}. \tag{2.8}$$

and

$$det(J_b) < 1 \Leftrightarrow \frac{ar(1-b)^2(r+ab)}{(r+ab^2)(a+r)} < 1. \tag{2.9}$$

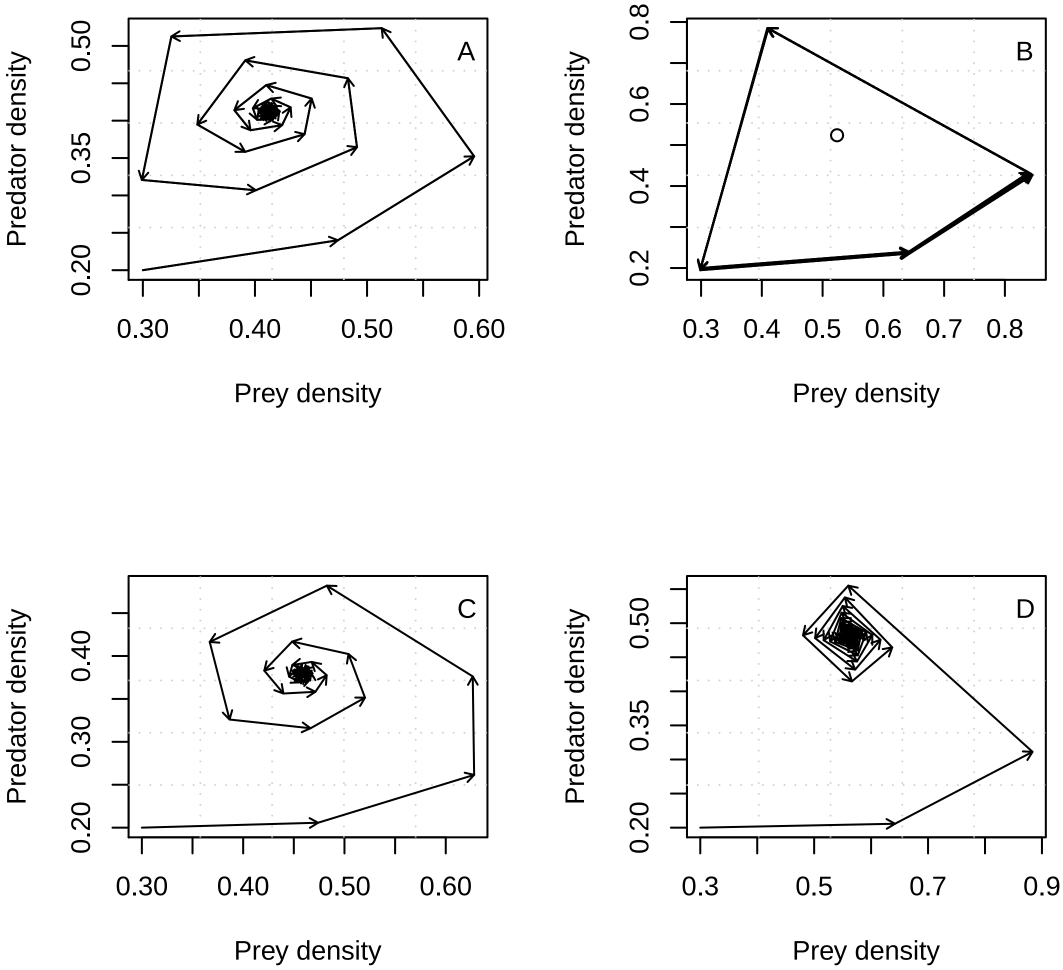

Fig. 1 The trajectories of predator and prey densities with and without Allee effect by using the initial conditions $N_0 = 0.3$, $P_0 = 0.2$ and by fixing $a = 2$. The graph in (A) (resp. (B)) indicates the solution of model (2.1) with $r = 1.4$ (resp. $r = 2.2$), however, the graph in (C) (resp. (D)) corresponds to model (2.4) when the predator is subject to the Allee effect with $r = 1.4$ (resp. $r = 2.2$) and $b = 0.08$.

Considering (2.8) and (2.9) one can get the following result:

Theorem 2.1 By condition (2.6), the positive equilibrium point (N_b^*, P_b^*) of the predator-prey system (2.4) is asymptotically stable if

$$2 - \frac{4(r+ab)}{r(r+ab^2)} < \frac{ar(1-b)^2(r+ab)}{(r+ab^2)(a+r)} < 1, \tag{2.10}$$

holds.

The following result is an immediate consequence of theorem 2.1

Corollary 2.2 By condition (2.6), the positive equilibrium point (N_b^*, P_b^*) of the predator-prey system (2.4) is unstable if and only if

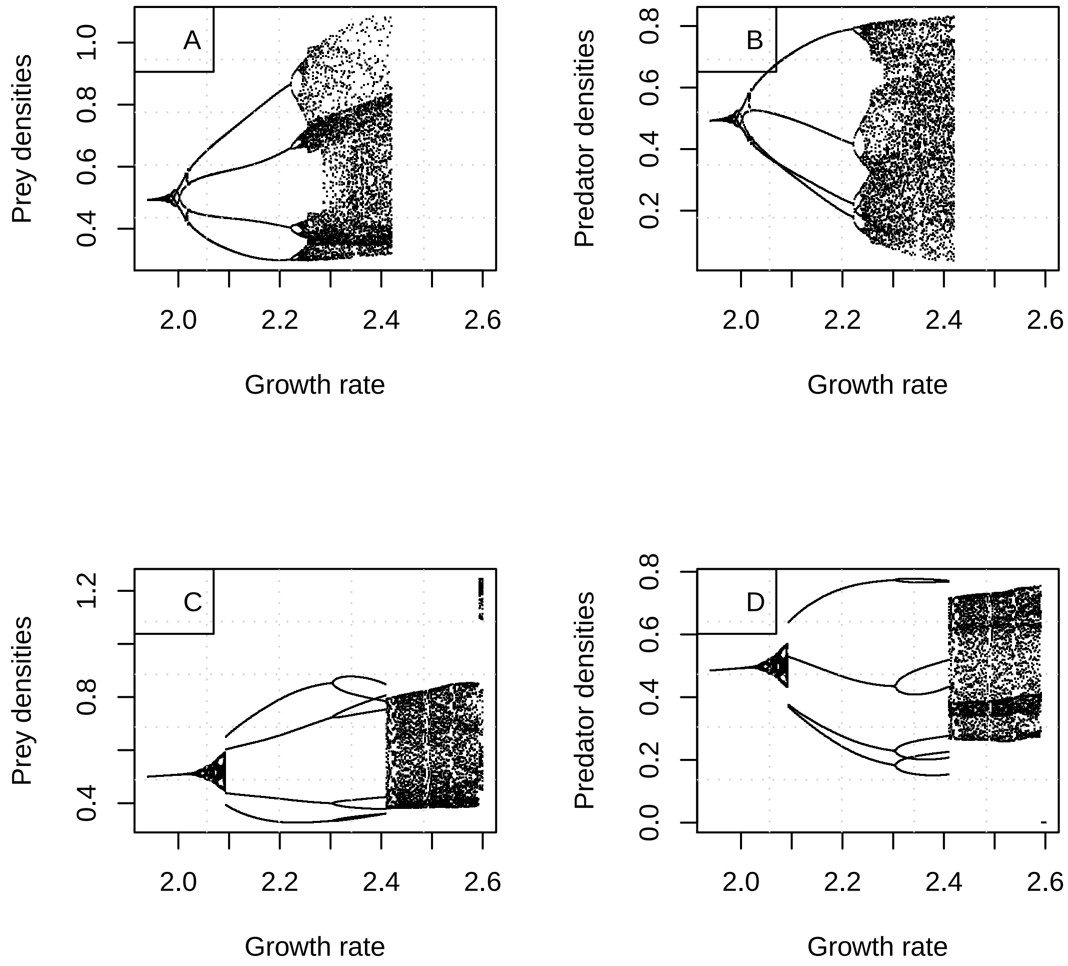

Fig. 2 Bifurcation diagrams of predator and prey densities in models (2.1)and (2.4) with the initial conditions $N_0 = 0.3$, $P_0 = 0.2$ and the parameter values $a = 2$, $b = 0.15$ and $r = 1.94{:}0.001{:}2.6$. The graps (C) & (D) are given by model (2.4) when the predator population is subject to the Allee effect while the others correspond to model (2.1).

$$2-\frac{4(r+ab)}{r(r+ab^2)} > \frac{ar(1-b)^2(r+ab)}{(r+ab^2)(a+r)} \text{ or } \frac{ar(1-b)^2(r+ab)}{(r+ab^2)(a+r)} > 1$$

holds.

Remarks: If we choose the Allee constant $b = 0$ (i.e., if there is no Allee on the predator population), then (2.10) reduces to (2.3) immediately. However, when $b < 1$, we see that the asymptotic stability of the equilibrium point (N_b^*, P_b^*) is stronger than that of (N_0^*, P_0^*) (Figure1). Furthermore, for some fixed parameters a, r, and b satisfying the conditions (2.3) and (2.10), we see that (N_b^*, P_b^*) is asymptotically stable while (N_0^*, P_0^*) is unstable.

Numerical Simulation

Now, we give the numerical simulations to verify our theoretical results. mainly, we present the graphs of the solutions N_t versus P_t (around the positive equilibrium point) for the systems (2.1) and (2.4), we illustrate the stabilizing effect of the Allee function. When we analyze the trajectories of the solutions around the positive equilibrium points for both models, we can easily see the stabilizing effect of the Allee function that we considered on the predator population in model (2.4).

In figure 1 we illustrate the trajectories of predator and prey densities in systems (2.1) and (2.4) by taking $a = 2$ and the initial conditions $N_0 = 0.3$, $P_0 = 0.2$. We use $r = 1.4$ in (1A) and (1C) while $r = 2.2$ in (1B) and (1D). Here (1A) and (1B) show the trajectories of predator and prey densities in model (2.1), however, (1C) and (1D) correspond to model (2.4) that is subject to the Allee effect by taking the same parameters as in (1A) and (1B). We see from (1A) and (1C) that when the predator population is subject to an Allee effect, the local stability of the equilibrium point increases and trajectory of the solution converges to the corresponding equilibrium point much faster. Furthermore, (1B) and (1D) represents that the corresponding equilibrium points move from instability to stability under Allee effect with Allee constant with $b = 0.08$. So we can conclude that the numerical simulations agree with the analytical results on the stabilizing effect on predator-prey system due to Allee effect on predator. This numerical procedure has been carried out by taking a wide range of parameter regions that supports our analytical observations.

Finally, Figure 2 indicates the bifurcation diagrams of models (2.1) and (2.4) with the initial conditions $N_0 = 0.3$, $P_0 = 0.2$ as above and the parameter values a = 2, b = 0.15 and r = 1.94:0.001:2.6. Fig. (2C) and (2D) show the bifurcations of prey and predator densities respectively of model (2.4), when the prey population is subject to the Allee effect. The graphs (2A) and (2B) correspond to the bifurcations of model (2.1). We observe that bifurcation value of model (2.1) is between the numbers 2.2 and 2.3, and for model (2.4) is between 2.4 and 2.5. These bifurcation diagrams are consistent with the analytical results and supports the mathematical analysis.

2.2 Allee effect II on prey only

In this section we consider the predator-prey system (2.1) as subject to an Allee effect II on prey population and analyze the following system:

$$\begin{aligned} N_{t+1} &= N_t + rN_t(1-N_t) - aN_tP_t\left(1+\tfrac{A_2}{N_t}\right), \\ P_{t+1} &= P_t + aP_t(N_t - P_t), \end{aligned} \tag{2.11}$$

where we take $(1+\frac{A_2}{N_t})$ as the Allee effect function and $A_2 > 0$ as the Allee constant for the Allee effect II of the prey. Such Allee effect is introduced by modifying the functional response, the functional response term will increase from aNP to $2aNP$ when N equals A_2. The system has three equilibrium points of the new system (2.11) as $(0,0)$, $(1,0)$, and (N_A^*, P_A^*), where

$$N_A^* = P_A^* = \frac{r-aA_2}{a+r} \tag{2.12}$$

Again, N_A^* and P_A^* will be positive under the condition

$$0 < A_2 < \frac{r}{a} \tag{2.13}$$

It is to be noted that, due to Allee effect in prey population the equilibrium population size decreases ($N_0^* > N_A^*$). By previous investigation, we can easily see that $(0,0)$ and $(1,0)$ are not asymptotically stable of the system (2.11).

Under condition (2.13), the predator-prey system (2.11) has unique positive equilibrium point (N_A^*,P_A^*). The bigger A_2 is, the stronger Allee II of the prey. If $A_2 = N_t$, then predation rate becomes double. It is clear that the point N_A^* (resp. P_A^*) is smaller than N_0^* (resp. P_0^*). In other words, both prey and predator densities at the equilibrium are decreased due to Allee effect II on prey population. It is also not difficult to show that the equilibrium is changed from stable to unstable. The trajectories of the prey and predator population are shown in Figure 3. At the point (N_A^*, P_A^*) the coefficient matrix of (2.11) after some simple computation turns out to be

$$J_A = \begin{bmatrix} 1-\alpha_A N_A^* & -\beta_A N_A^* \\ aN_A^* & 1-aN_A^* \end{bmatrix},$$

where

$$\alpha_A = \frac{r^2 - arA_2 - a^2A_2}{r-aA_2} \text{ and } \beta_A = \frac{ar(1+A_2)}{r-aA_2}.$$

Then we have the corresponding characteristic equation to the matrix J_A as follows:

$$P_A(\lambda) = \lambda^2 - tr(J_A)\lambda + det(J_A) = 0, \tag{2.14}$$

where

$$tr(J_A) = 2-(a+\alpha_A)N_A^*$$

and

$$det(J_A) = 1-r+2aA_2+\frac{a(r-aA_2^2)^2}{a+r}.$$

Similarly using Jury conditions and after simple calculation we have the following theorem:

Theorem 2.3 By condition (2.13), the positive equilibrium point (N_A^*, P_A^*) of the predator-prey system (2.11) is asymptotically stable if

$$2-\tfrac{4}{r-2aA_2} < \tfrac{a(r-aA_2)}{a+r} < 1, \tag{2.15}$$

holds.

The next result follows from Theorem 2.3 immediately.

Corollary 2.4 By condition (2.13), the positive equilibrium point (N_A^*, P_A^*) of the predator-prey system (2.11) is unstable if and only if

$$2-\frac{4}{r-2aA_2} > \frac{a(r-aA_2)}{a+r} \text{ or } \frac{a(r-aA_2)}{a+r} > 1$$

holds.

Remark If we choose the Allee constant $A_2 = 0$ (i.e., if there is no Allee effect on the prey population), then (2.15) reduces to (2.3) immediately. However, when $0 < A_2 < \frac{r}{a}$, we see that the asymptotic stability of the equilibrium point (N_A^*, P_A^*) is weaker than that of (N_0^*, P_0^*) (Figure 3). Furthermore, for some fixed parameters a, r, and A_2 satisfying the conditions (2.3) and (2.15), we see that (N_A^*, P_A^*) is unstable while (N_0^*, P_0^*) is stable.

Numerical Simulation

Now, we perform the numerical simulations to verify our theoretical results by using R software package. We present the graphically of the solutions N_t versus P_t (around the positive equilibrium point) for systems (2.11) and this illustrates the destabilizing effect of the Allee function II on prey population. By analyzing the trajectories of the solutions around the positive equilibrium points, describing affect of type II Allee on prey (model (2.11)) can be easily verified.

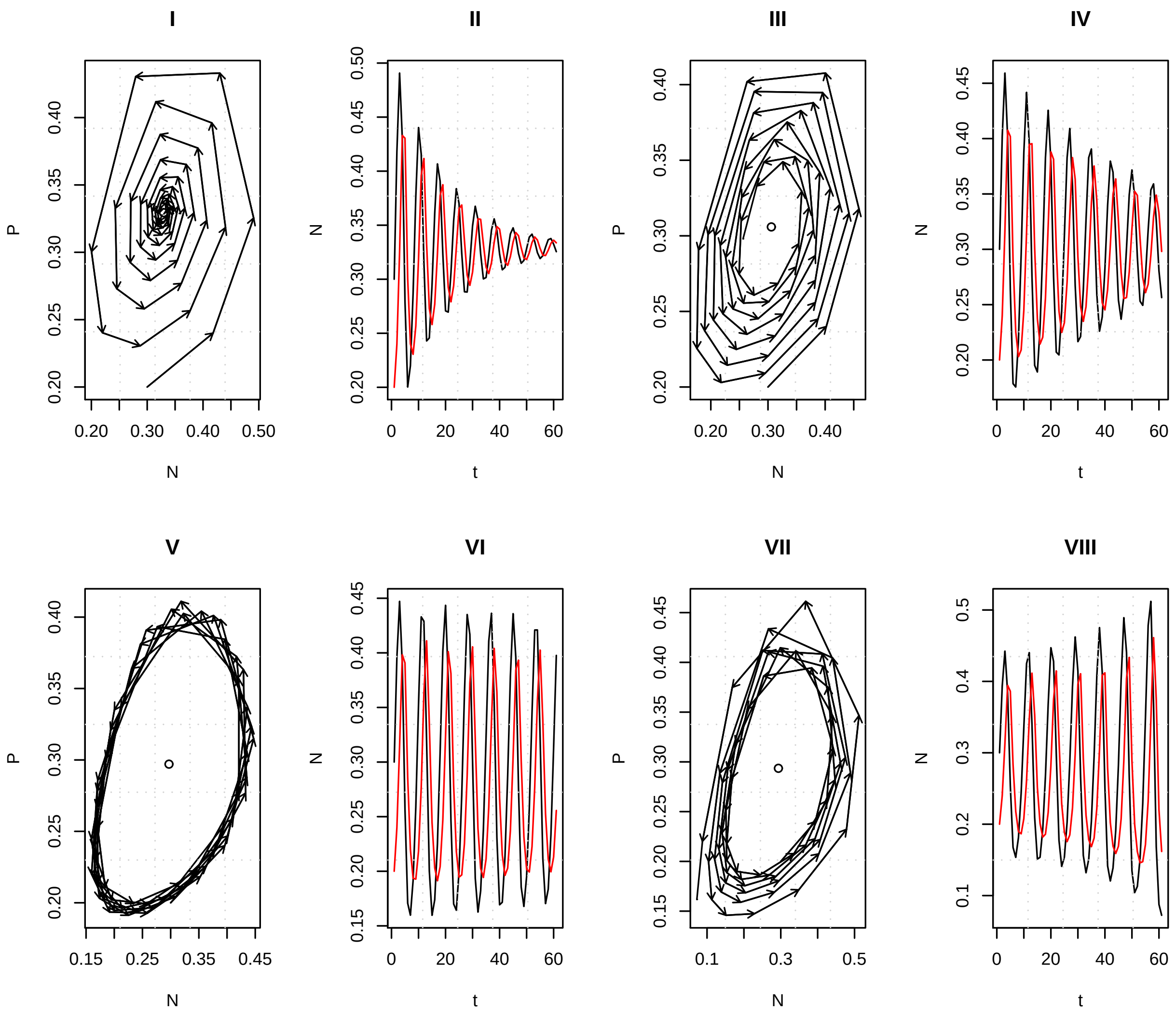

Fig. 3 The trajectories of predator-prey densities with Allee effect and prey population with time by using initial conditions $N_0 = 0.3$, $P_0 = 0.2$ and by fixing $r = 1.4$, $a = 2$ of model (2.11). The graph (I), (III), (V), (VII) indicates the solution of predator versus. prey with Allee $A_2 = 0.14, 0.18, 0.195$ and 0.201 respectively, however, the graph in (II), (IV), (VI), (VIII) indicates prey densities with time with respect to same Allee respectively.

In figure 3 we illustrate the trajectories of predator-prey densities and prey density with time in system (2.11) by taking $a = 2$, $r = 1.4$ and the initial conditions $N_0 = 0.3$, $P_0 = 0.2$. We use $A_2 = 0.14$ in 3(I) and 3(II), $A_2 = 0.18$ in 3(III) and 3(IV), $A_2 = 0.195$ in

3(V) and 3(VI) while $A_2 = 0.201$ in 3(VII) and 3(VIII).

Here figure 3(I, III, V and VII) show the trajectories of predator and prey densities in model (2.11), however, figure 3(II, IV, VI and VIII) correspond to trajectories of prey in model (2.11). We see from figures that when the prey population is subject to an Allee effect II, the local stability of the equilibrium point decreases. Furthermore, 3(I) and 3(III) presents that the corresponding equilibrium points move from stability to damping oscillation, 3(V) shows neutral stability and finally 3(VII) shows instability under the Allee effect. So we can conclude that the numerical simulations agree with the analytical results on the destabilizing effect of the Allee function that we incorporate on prey population.

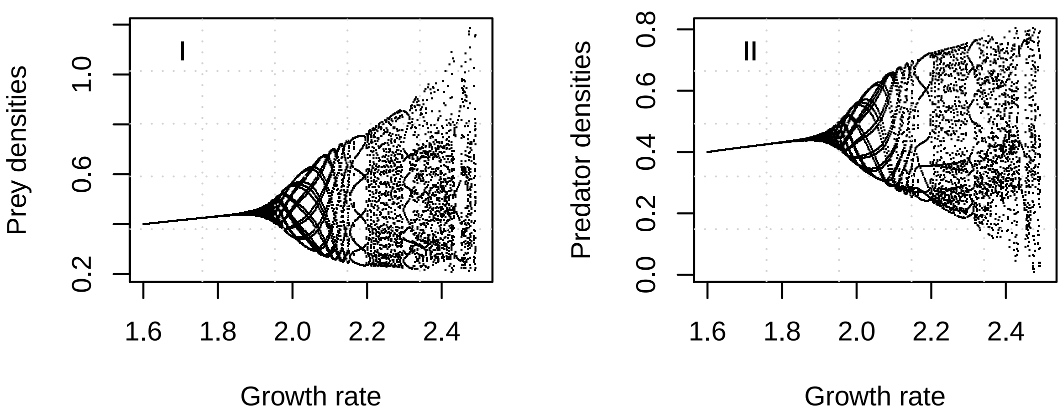

Fig. 4 Bifurcation diagrams of predator and prey densities in model (2.11) with initial conditions $N_0 = 0.3$, $P_0 = 0.2$, and the parameter values $a = 2$, $A_2 = 0.08$ and $r = 1.6 : 0.001 : 2.5$

Finally, Figure 4 indicates the bifurcation diagrams of model (2.11) with the initial conditions N_0 = 0.3, P_0 = 0.2 as above and the parameter values a = 2, A_2 = 0.08 and r = 1.6:0.001:2.5. Fig. 4(I) and 4(II) show the bifurcations of prey and predator densities of model (2.11), respectively, when the prey population is subject to Allee effect. We observe that bifurcation value of model (2.1) is between the numbers 2.2 and 2.3, (Figure 1, A and B) and for model (2.11) is between 1.8 and 2. These bifurcation diagrams are consistent with the analytical results and supports the mathematical analysis.

2.3 Allee effect on both predator and prey

In this section we consider the predator prey system (2.1) as subject to Allee effect on both prey and predator and analyze the following system:

$$\begin{aligned} N_{t+1} &= N_t + rN_t(1-N_t)\frac{N_t}{u+N_t} - aN_tP_t, \\ P_{t+1} &= \quad P_t + aP_tN_t\left(\frac{P_t}{b+P_t}\right) - P_t, \end{aligned} \tag{2.16}$$

where we take $\frac{N_t}{u+N_t}$ as the Allee effect function on prey population and $u > 0$ as the Allee constant on prey. The bigger u is, the stronger Allee effect of the prey. Similarly, $\frac{P_t}{b+P_t}$ as the Allee effect function on predator population and $b > 0$ as the Allee constant on predator. We have four equilibrium points of the new system (2.16) as $(0,0)$, $(1,0)$, (N_u^{*1}, P_b^{*1}), and (N_u^{*2}, P_b^{*2}), where
$N_u^{*1} = \mu$, $P_b^{*1} = \mu - b$, $N_u^{*2} = \mu_1$ and $P_b^{*2} = \mu_1 - b$, where

$$\mu = \frac{(r-au+ab)+\sqrt{(r-au+ab)^2+4(a+r)abu}}{2(a+r)},$$

$$\mu_1 = \frac{(r-au+ab)-\sqrt{(r-au+ab)^2+4(a+r)abu}}{2(a+r)}.$$

For positive equilibrium we consider

$$b < 1. \tag{2.17}$$

By using condition (2.17) it is easy to verify that (N_u^{*1}, P_b^{*1}) is the only positive equilibrium point as $\mu_1 < 0$ implies both N_u^{*2} and P_b^{*2} are negative.

By previous investigation, we can easily see that $(0,0)$ and $(1,0)$ are not asymptotically stable equilibrium points of the system (2.16). The jacobian matrix of the system (2.16) at the equilibrium point (N_u^{*1}, P_b^{*1}) is

$$J = \begin{bmatrix} 1+rN_u^{*1}\frac{2u+N_u^{*1}-3uN_u^{*1}-2(N_u^{*1})^2}{(u+N_u^{*1})^2}-aP_b^{*1} & -aN_u^{*1} \\ \frac{a(P_b^{*1})^2}{b+P_b^{*1}} & 1+\frac{aN_u^{*1}P_b^{*1}(2b+P_b^{*1})}{(b+P_b^{*1})^2)}-2aP_b^{*1} \end{bmatrix}.$$

Then we have the corresponding characteristic equation to the matrix J as follows:

$$P(\lambda) = \lambda^2 - tr(J)\lambda + det(J) = 0,$$

where

$$tr(J) = 2 + r_1 \text{ and } det(J) = 1 + r_1 + r_2$$

where

$$r_1 = rN_u^{*1}\frac{2u+N_u^{*1}-3uN_u^{*1}-2(N_u^{*1})^2}{(u+N_u^{*1})^2}+\frac{aN_u^{*1}P_b^{*1}(2b+P_b^{*1})}{(b+P_b^{*1})^2}-3aP_b^{*1} \tag{2.18}$$

$$\begin{aligned} r_2 = -2arN_u^{*1}P_u^{*1}\frac{\left(2u-3uN_u^{*1}+N_u^{*1}-2(N_u^{*1})^2\right)}{(u+N_u^{*1})^2}-a^2N_u^{*1}(P_b^{*1})^2\frac{(2b+P_b^{*1})}{(b+P_b^{*1})^2}+2a^2(P_b^{*1})^2 \\ +ar(N_u^{*1})^2P_b^{*1}\frac{\left(2u-3uN_u^{*1}+N_u^{*1}-2(N_u^{*1})^2\right)(2b+P_b^{*1})}{\left(u+N_u^{*1}\right)^2(b+P_b^{*1})} \end{aligned} \tag{2.19}$$

By using the same calculation using Jury conditions we have the following theorem:

Theorem 2.5 By condition (2.17), the positive equilibrium point (N_u^{*1}, P_b^{*1}) of the predator-prey system (2.16) is asymptotically stable if

$$2+\frac{4}{r_1} < -\frac{r_2}{r_1} < 1. \tag{2.20}$$

holds.

The following result is an immediate consequence of theorem 2.5.

Corollary 2.6 By condition (2.17), the positive equilibrium point (N_u^{*1}, P_b^{*1}) of the predator-prey system (2.16) is unstable if and only if

$$2+\frac{4}{r_1} > -\frac{r_2}{r_1} \text{ or } -\frac{r_2}{r_1} > 1$$

holds.

Remark: If we choose the Allee constants $u = 0$ and $b = 0$, then 2.20 reduces to 2.3 immediately. However, when $b < 1$, and $0 < u < \frac{r}{a}$ we see that the asymptotic stability of the equilibrium point (N_u^{*1}, P_b^{*1}) is stronger than that of (N_0^*, P_0^*) (Figure 1 and 5). Furthermore, for some fixed parameters a, r, b and u satisfying the conditions 2.3 and 2.20, we see that (N_u^{*1}, P_b^{*1}) is asymptotically stable while (N_0^*, P_0^*) is unstable.

Numerical Simulation

We present the graphs of the solutions N_t versus P_t (around the positive equilibrium point) for the predator-prey system (2.16), we illustrate the stabilizing effect of the Allee function. When we analyze the trajectories of the solutions around the positive equilibrium points for both models, we can easily see the stabilizing effect of the Allee function that we impose on both the population by model (2.16).

In figure 5 we illustrate the trajectories of predator and prey densities in system (2.16) by taking $a = 2$, $r = 1.4$ and the initial conditions $N_0 = 0.3$, $P_0 = 0.2$. We use $u = 0.06$, $b = 0.08$ in (5a) and (5b) while $b = 4$ in (5c) and (5d). Here (5a) and (5c) show the trajectories of predator and prey densities in model (2.16), however, (5c) and (5d) correspond to predator and prey densities over time. We see that when the predator and prey population is subject to an Allee effect, the local stability of the equilibrium point increases and trajectory of the solution approximates to the corresponding equilibrium point much faster. Furthermore, (5c) presents that the corresponding equilibrium points move from instability to stability under Allee effects. Thus the numerical simulations agree with the analytical results on the stabilizing effect of the Allee function that we incorporate on predator and prey populations. By varying the Allee parameters u and b a more complex dynamical behavior may be obtained.

Finally, Figure 6 indicates the bifurcation diagrams of model (2.16) with the initial conditions $N_0 = 0.3$, $P_0 = 0.2$ as above and the parameter values $a = 2$, $b = 0.08$, $u = 0.06$ and $r = 2.5{:}0.001{:}3.2$. Fig. (6a) and (6b) show the bifurcations of prey and predator densities of model (2.16), respectively, when both population is subject to Allee effect. We observe that bifurcation value of model (2.1) is between the numbers 2.2 and 2.3 (Figure 1, A and B), and for model (2.16) is between 2.6 and 2.7. These bifurcation diagrams are consistent with the analytical results and supports the mathematical analysis.

3 Conclusion

Since the pioneering work by Lotka & Volterra (24; 25), the prey-predator system or competitive inter-specific species interaction has become one of the central theories in population dynamics. With its wide range of application in understating the community and

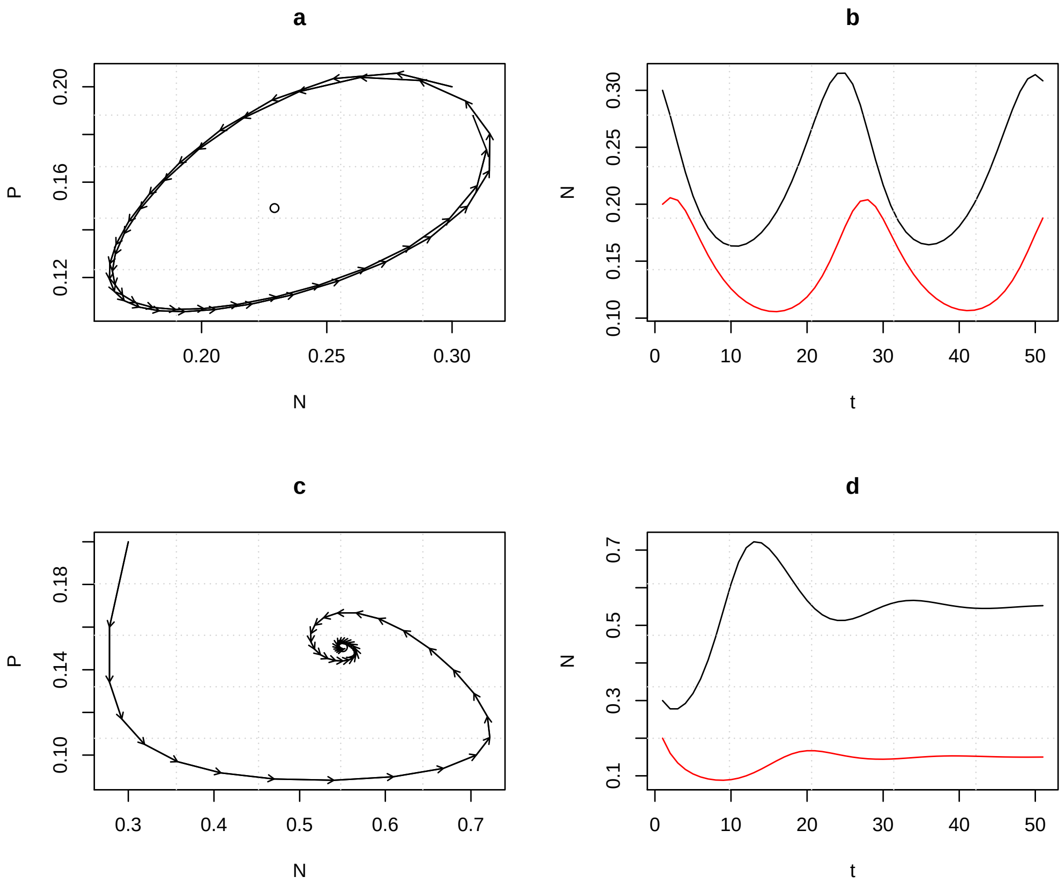

Fig. 5 The trajectories of predator-prey densities with Allee effect and prey-predator population with time by using initial conditions $N_0 = 0.3$, $P_0 = 0.2$ and by fixing $r = 1.4$, $a = 2$ of model (2.16). The graphs 5(a) and 5(c) indicates the solution of predator versus prey with Allee effects $u = 0.6$, however, $b = 0.08$, and 0.4 respectively. Again the graphs 5(b) and 5(d) indicates predator and prey densities with time with respect to same Allee respectively.

population level consequences of inter-specific species dynamics, it plays a crucial role in both theoretical and applied ecology. In this paper, we have selected classical logistic model for describing predator-prey systems and introduced Allee effects into predator, prey and both predator and prey, respectively. By combining mathematical analysis and numerical simulations, we have shown that, Allee effect plays not only a stabilizing role in discrete-time predator-prey system but, it may act as a destabilizing force as well.

After analyzing the trajectories of the solutions around the equilibrium points of the models, both (6) and (20) have found the stabilizing effect Allee mechanism due to mating limitation. Our analysis reveals that when Allee is active on prey and it is of type-II, the equilibrium may be changed from asymptotically stable to unstable or the system may take much longer time to reach stable state. But systems 2.1 and 2.3 have similar implications as observed by (20). The stability behavior is mainly dependent on how Allee effect is incorporated on prey or predator. But the influences of the Allee on prey and predator, the systems' stability and the magnitude of equilibrium may vary according to the model assumption, construction and magnitude of the Allee constants. It would be very interesting to study the system under delayed framework. One can also think of a plausible extension of the system with

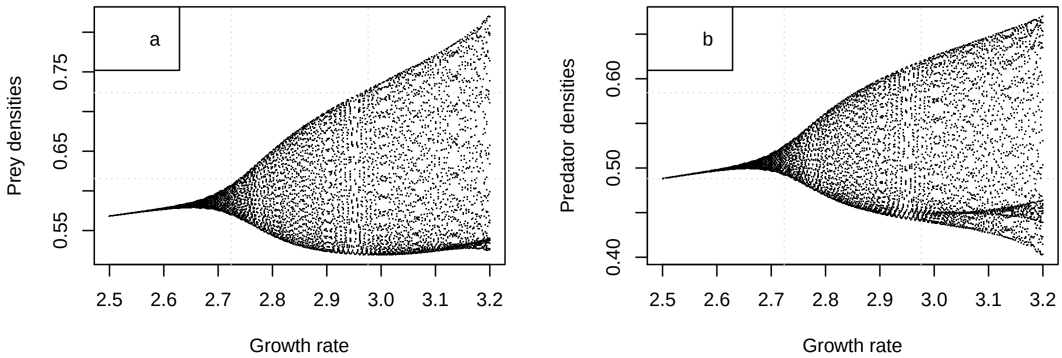

Fig. 6 Bifurcation diagrams of predator and prey densities in model (2.16) with initial conditions $N_0 = 0.3$, $P_0 = 0.2$,and the parameter values $a = 2$, $u = 0.06$, $b = 0.08$ and $r = 2.5 : 0.001 : 3.2$.

alternative food for predator. A further deeper analysis in future would demand the enhancement of significant ecological implications for better understanding of prey-predator dynamics.

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