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Dynamics of a three species modified Leslie-Gower food-web system with switching predator

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Abstract. *In this paper, we investigate the dynamics of a three species food-web system. The system consisting of a logistically growing prey and two predators. One of the predators is a generalist predator with modified Leslie-Gower functional response due to additional food. The second predator is switching between the prey and its generalist predator. Stability analysis has been carried out for all possible axial, boundary and interior equilibrium points. In has been shown that the system undergoes the transcritical bifurcation about two axial points. All the boundary points are shown to be globally asymptotically stable in the positive quadrant of their respective planes. The system is shown to be persist at stable interior point as well as in the form of periodic solution. At the end, numerical simulations have been performed to investigate the Hopf, generalized-Hopf and Bogdanov-Takens bifurcations about the positive interior point.*

Key words: Modified Leslie Gower, Switching, Persistence, Hopf bifurcation

1 Introduction

Predator- Prey interactions are well- studied in ecology. One significant component of prey predator relationship is the predator's feeding rate upon prey. Holling derived mainly three types of functional responses, namely Holling- type I, II and III. Holling type II functional response describes the average feeding rate of predator when the predator spends time for searching and handling. This function is widely used by researchers to study three species food-web models [8, 9]. Another type of functional response which was

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given by Leslie, has also studied by some authors [2, 3, 15].

In ecosystem, several species are present. In this situation, predator has the tendency to feed upon more than one prey due to lack of its preferred food. Therefore, predator's ability to hunt different prey species when its favourite prey species has declined, is called switching. Such type of predators are called switching predators. The term switching was firstly studied by the ecologist Murdoch (1969). This concept has created interest among many mathematicians and switching effect is analyzed using mathematical models in [1, 4, 5, 6].

Khan et al. [1] considered a three species switching model with group defence. They have studied the thresholds, equilibria, stability and Hopf bifurcation. Mukhopadhyay and Bhattacharyya [4] have considered the ecological model with one predator, switching over two logistically growing independent prey species. The Hopf bifurcation analysis has been carried out. The existence of stable limit cycle has also been shown through numerical simulation. Tansky [5] investigated the two prey and one predator system having switching property. Further, Ajraldi [6] extended the Tansky model and considered the dynamical system with one switching predator and two prey species with intraspecific and interspecific competitions. The interior equilibrium point has proven to be unconditionally stable.

In this paper, three species model is proposed with one logistically growing prey and two predators of which one is a generalist predator. The switching behavior of second predator between the top prey and its generalist predator is considered.

2 Model formulation

The dynamics of three species modified Leslie-Gower food-web with switching predator is governed by the following system of non-linear differential equations:

$$\begin{aligned}
\frac{dX}{dT} &= r_1 X\left[1-\frac{X}{K}\right]-\frac{a_1XY}{b_1+X}-\frac{\alpha_{11}XZ}{\left(1+\frac{Y}{X}\right)}\\
\frac{dY}{dT} &= Y\left[r_2-\frac{a_2Y}{b_2+X}\right]-\frac{\alpha_{12}YZ}{\left(1+\frac{X}{Y}\right)}\\
\frac{dZ}{dT} &= -e_1Z+\frac{\beta_{11}XZ}{\left(1+\frac{Y}{X}\right)}+\frac{\beta_{12}YZ}{\left(1+\frac{X}{Y}\right)}
\end{aligned} \tag{2.1}$$

The non-negative initial conditions are: $X(0)\geq 0, Y(0)\geq 0, Z(0)\geq 0$

In system (2.1), X is the density of logistically growing prey species with intrinsic growth rate r_1 and carrying capacity K, Y is the density of generalist predator with modified Leslie- Gower functional response. It is predating over X and an additional source of food is being considered for it. The switching predator Z is predating over X and Y with functional responses $\frac{\alpha_{11}Z}{\left(1+\frac{Y}{X}\right)}$ and $\frac{\alpha_{12}Z}{\left(1+\frac{X}{Y}\right)}$ respectively. The denominator terms represent the switching behavior. When X is in abundance, Z takes food from it and when X is in short supply, Z switches over to Y. Predation rates and conversion rates of the switching predator are α_{1i} and $\beta_{1i}; (i=1,2)$ respectively and e_1 is the mortality rate of predator Z.

The following non-dimensional variables and parameters are introduced:

$t=r_1T,\ x=\frac{X}{K},\ y=\frac{Y}{K},\ z=\frac{a_1Z}{r_1K}$

$w_1=\frac{a_1}{r_1},\ \ w_2=\frac{a_2}{r_1},\ \ w_3=\frac{e_1}{r_1},\ \ r=\frac{r_2}{r_1},\ A_1=\frac{b_1}{K},$

$A_2=\frac{b_2}{K},\ \alpha_1=\frac{\alpha_{11}K}{a_1},\ \alpha_2=\frac{\alpha_{12}K}{a_1}, \beta_1=\frac{\beta_{11}K}{r_1},\ \beta_2=\frac{\beta_{12}K}{r_1},$

Accordingly, the system (2.1) will take the following non-dimensional form:

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x)-\frac{w_1xy}{x+A_1}-\frac{\alpha_1x^2z}{x+y}=F_1(x,y,z)\\
\frac{dy}{dt} &= y\left[r-\frac{w_2y}{x+A_2}\right]-\frac{\alpha_2y^2z}{x+y}=F_2(x,y,z)\\
\frac{dz}{dt} &= -w_3z+\frac{\beta_1x^2z}{x+y}+\frac{\beta_2y^2z}{x+y}=F_3(x,y,z)
\end{aligned} \tag{2.2}$$

$x(0)\geq 0, y(0)\geq 0, z(0)\geq 0$

3 Preliminaries

The interaction functions $F_i(i=1,2,3)$ are continuous and have continuous partial derivatives in the state space R_+^3. Hence the solution of the system with non-negative initial conditions exists and it is unique.
The positivity of the solution of the system (2.2) can be easily proved by Nagumo's Theorem [14] and boundedness of the system is established in the following theorem.

Theorem 1. The solution of the system (2.2) initiate in R_+^3 is bounded.

Proof. From the first equation of (2.2), we have

$$\frac{dx}{dt} \leq x(1-x), \ \Rightarrow x(t) \leq \frac{1}{1+\frac{1}{C}e^{-t}} \ \forall t \geq 0,$$

where $\dfrac{1}{C} \leq \dfrac{1}{x_0} - 1$. Hence for large value of time

$$x(t) \leq 1, \ \forall t \geq 0. \tag{3.1}$$

Let $\phi(t) = \dfrac{x(t)}{\alpha_1} + \dfrac{\beta_2 y(t)}{\alpha_2\beta_1} + \dfrac{z(t)}{\beta_1}$, $\phi(0) = \phi_0 \geq 0$
It is easy to verify that

$$\frac{d\phi}{dt} + w_3\phi \leq \frac{x}{\alpha_1}(1+w_3-x) + \frac{\beta_2 y}{\alpha_2\beta_1}\left(r + w_3 - \frac{w_2 y}{A_2+1}\right) \leq \frac{(1+w_3)^2}{4\alpha_1} + \frac{\beta_2(r+w_3)^2(A_2+1)}{4w_2\alpha_2\beta_1} \tag{3.2}$$

We can define a positive constant M_1, such that

$$M_1 = \frac{(1+w_3)^2}{4\alpha_1} + \frac{\beta_2(r+w_3)^2(A_2+1)}{4w_2\alpha_2\beta_1} > 0$$

Equation (3.2) can be written as

$$\frac{d\phi}{dt} + w_3\phi \leq M_1$$

After applying the theory of differential inequality, we obtain

$$0 < \phi \leq \frac{M_1}{w_3}(1-e^{-w_3 t}) + e^{-w_3 t}\phi_0$$

$$\Rightarrow 0 < \phi \leq \tfrac{M_1}{w_3}$$

Thus $\phi(t) = \dfrac{x(t)}{\alpha_1} + \dfrac{\beta_2 y(t)}{\alpha_2\beta_1} + \dfrac{z(t)}{\beta_1} \leq \dfrac{M_1}{w_3}$.
Hence all the species are bounded in R_+^3.

4 Existence and feasibility of Equilibrium points

1. The trivial point $E_0 = (0,0,0)$ and axial equilibrium points $E_1 = (1,0,0)$ and $E_2 = \left(0, \dfrac{rA_2}{w_2}, 0\right)$ always exist.
2. In the absence of switching predator, the point in $x-y$ plane $E_{12} = (\hat{x}, \hat{y}, 0)$ is obtained by solving the following equations:

$$\begin{aligned} (1-x) - \frac{w_1 y}{x+A_1} &= 0 \\ r - \frac{w_2 y}{x+A_2} &= 0 \end{aligned} \tag{4.1}$$

It can be easily seen that $\hat{x}$ satisfies the equation:

$$w_2\hat{x}^2 + \hat{x}(w_2A_1 + w_1 r - w_2) + (w_1 r A_2 - w_2 A_1) = 0$$

Thus
$\hat{x}_{\pm} = \frac{1}{2w_2}(-(w_2A_1 + w_1r - w_2) \pm \Delta^{\frac{1}{2}})$,
where $\Delta = (w_2A_1 + w_1r - w_2)^2 - 4w_2(w_1rA_2 - w_2A_1)$, Δ is nonnegative if

$$\frac{w_1r}{A_1} < \frac{w_2}{A_2} \tag{4.2}$$

Under (4.2), $\hat{x}_+ > 0$ and $\hat{x}_- < 0$
The boundary equilibrium point $E_{12} = (\hat{x}, \hat{y}, 0)$ exists under (4.2), where

$$\hat{x} = \frac{1}{2w_2}(-(w_2A_1 + w_1r - w_2) + \Delta^{\frac{1}{2}}), \ \hat{y} = \frac{r(\hat{x} + A_2)}{w_2}$$

3. In the absence of Leslie-Gower predator, the equilibria in $x - z$ plane is given by $E_{13} = (\tilde{x}, 0, \tilde{z})$ where

$$\tilde{x} = \frac{w_3}{\beta_1}, \ \tilde{z} = \frac{\beta_1 - w_3}{\alpha_1\beta_1} \tag{4.3}$$

The equilibrium point E_{13} exists, provided:

$$\beta_1 > w_3 \tag{4.4}$$

4. In the absence of logistically growing prey the point in $y - z$ plane is given by $E_{23} = (0, \bar{y}, \bar{z})$, where

$$\bar{y} = \frac{w_3}{\beta_2}, \ \bar{z} = \frac{1}{\alpha_2A_2}[rA_2 - w_2\bar{y}] \tag{4.5}$$

The boundary equilibrium point E_{23} exists, provided:

$$\frac{w_3}{\beta_2} < \frac{rA_2}{w_2} \tag{4.6}$$

5. The interior equilibrium point is obtained as $E^* = (x^*, y^*, z^*)$ where

$$x^* = \frac{w_3P_e(1+P_e)}{\beta_1P_e^2 + \beta_2}, \ y^* = \frac{w_3(1+P_e)}{\beta_1P_e^2 + \beta_2}$$
$$z^* = \frac{(1+P_e)}{\alpha_1P_e}\left[(1 - x^*) - \frac{w_1y^*}{x^* + A_1}\right] \ \textit{or equivalently}$$
$$z^* = \frac{(1+P_e)}{\alpha_2}\left[r - \frac{w_2y^*}{x^* + A_2}\right]$$

where $P_e = \frac{x^*}{y^*}$

5 Analysis

5.1 Local stability analysis

In this section, stability analysis is carried out about all possible equilibrium points. The Jacobian matrix about an equilibrium point $E : (x, y, z)$ is given as

$$J_E = (a_{ij})_{3x3} = \begin{bmatrix} 1 - 2x - \frac{w_1A_1y}{(x+A_1)^2} - \frac{\alpha_1xz(x+2y)}{(x+y)^2} & -\frac{w_1x}{x+A_1} + \frac{\alpha_1x^2z}{(x+y)^2} & -\frac{\alpha_1x^2}{x+y} \\ \frac{w_2y^2}{(x+A_2)^2} + \frac{\alpha_2y^2z}{(x+y)^2} & r - \frac{2w_2y}{x+A_2} - \frac{\alpha_2yz(y+2x)}{(x+y)^2} & -\frac{\alpha_2y^2}{x+y} \\ \frac{\beta_1xz(x+2y)}{(x+y)^2} - \frac{\beta_2y^2z}{(x+y)^2} & -\frac{\beta_1x^2z}{(x+y)^2} + \frac{\beta_2yz(y+2x)}{(x+y)^2} & \frac{(\beta_1x^2 + \beta_2y^2)}{x+y} - w_3 \end{bmatrix}$$

1) The eigenvalues at E_0 are $1, r$ and $-w_3$. Thus trivial equilibrium point is saddle having stable manifold in z-direction.

2) The eigenvalues at E_1 are $-1, r$ and $\beta_1 - w_3$. The axial point will always be a saddle point. There will be a possibility of transcritical bifurcation for

$$w_3 = \beta_1 \tag{5.1}$$

3) The eigenvalues at E_2 are computed as

$$\lambda_1 = 1 - \frac{rw_1A_2}{A_1w_2},\ \lambda_2 = -r \text{ and } \lambda_3 = -w_3 + \frac{rA_2\beta_2}{w_2}$$

The axial point is locally asymptotically stable, provided

$$\frac{w_1r}{A_1} > \frac{w_2}{A_2} \text{ and } \frac{w_3}{\beta_2} > \frac{rA_2}{w_2} \tag{5.2}$$

The point E_2 will be unstable when one of the conditions of (5.2) is violated. The transcritical bifurcation may occur for

$$w_1 = \frac{A_1w_2}{rA_2} \text{ or } w_3 = \frac{rA_2\beta_2}{w_2}$$

4) The coefficients of Jacobian matrix at E_{12} are evaluated as

$$a_{11} = \hat{x}\left[-1 + \frac{w_1\hat{y}}{(\hat{x}+A_1)^2}\right],\ a_{12} = -\frac{w_1\hat{x}}{\hat{x}+A_1} < 0,\ a_{13} = -\frac{\alpha_1\hat{x}^2}{\hat{x}+\hat{y}} < 0$$
$$a_{21} = \frac{r^2}{w_2} > 0,\ a_{22} = -r < 0,\ a_{23} = -\frac{\alpha_2\hat{y}^2}{\hat{x}+\hat{y}} < 0$$
$$a_{31} = 0,\ a_{32} = 0,\ a_{33} = -w_3 + \frac{(\beta_1\hat{x}^2 + \beta_2\hat{y}^2)}{\hat{x}+\hat{y}}$$

The characteristic equation is:

$$\lambda^3 + a_0\lambda^2 + a_1\lambda + a_2 = 0$$

where

$$\begin{aligned}
a_0 &= -(a_{11} + a_{22} + a_{33}) \\
a_1 &= a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{12}a_{21} \\
a_2 &= a_{12}a_{21}a_{33} - a_{11}a_{22}a_{33} \\
a_0a_1 - a_2 &= -a_{11}(a_{33})^2 - (a_{11}+a_{22})(a_{11}a_{22} + a_{11}a_{33} - a_{12}a_{21}) - a_{22}a_{33}(a_{11}+a_{22}+a_{33})
\end{aligned}$$

For the local stability of the point E_{12}, the Routh-Hurwitz criteria gives

$$\hat{y} < \frac{(\hat{x}+A_1)^2}{w_1} \tag{5.3}$$

$$w_3 > \frac{(\beta_1\hat{x}^2 + \beta_2\hat{y}^2)}{\hat{x}+\hat{y}} \tag{5.4}$$

5) The coefficients of Jacobian matrix at E_{13} are

$$a_{11} = -\tilde{x} < 0,\ a_{12} = 1 - \tilde{x}\left[1 + \frac{w_1}{\tilde{x}+A_1}\right],\ a_{13} = -\alpha_1\tilde{x} < 0$$
$$a_{21} = 0,\ a_{22} = r > 0,\ a_{23} = 0$$
$$a_{31} = \beta_1\tilde{z} > 0,\ a_{32} = -\beta_1\tilde{z} < 0,\ a_{33} = 0$$

The characteristic equation is:

$$\lambda^3 + a_0\lambda^2 + a_1\lambda + a_2 = 0$$

where

$$\begin{aligned}
a_0 &= -(a_{11} + a_{22}) = \tilde{x} - r \\
a_1 &= a_{11}a_{22} = -r\tilde{x} \\
a_2 &= a_{13}a_{22}a_{31} = -\frac{w_3}{\beta_1}(\beta_1 - w_3) \\
a_0a_1 - a_2 &= -(a_{11})^2a_{22} - a_{11}(a_{22})^2 + a_{11}a_{13}a_{31}
\end{aligned}$$

By Routh-Hurwitz criteria the point E_{13} will be locally asymptotically stable if $a_0 > 0$, $a_2 > 0$ and $a_0a_1 - a_2 > 0$. Under the condition (4.4), $a_2 < 0$. Hence, the point $E_{13} = (\tilde{x}, 0, \tilde{z})$ will be unstable.

It is to be noted that, the planar point E_{13} becomes axial point E_1 under the condition (5.1).

6) The coefficients of the Jacobian matrix at E_{23} are

$$a_{11} = 1 - \frac{w_1}{A_1}\bar{y},\ a_{12} = 0,\ a_{13} = 0$$
$$a_{21} = \left[r + \frac{w_2 w_3}{A_2\beta_2}\left(\frac{\bar{y}}{A_2} - 1\right)\right],\ a_{22} = -\frac{w_2}{A_2}\bar{y} < 0,\ a_{23} = -\alpha_2\bar{y} < 0$$
$$a_{31} = -\frac{\beta_2}{A_2\alpha_2}(rA_2 - w_2\bar{y}) < 0,\ a_{32} = \frac{\beta_2}{A_2\alpha_2}(rA_2 - w_2\bar{y}) > 0,\ a_{33} = 0$$

The characteristic equation is:

$$\lambda^3 + a_0\lambda^2 + a_1\lambda + a_2 = 0$$

where

$$\begin{aligned} a_0 &= -(a_{11} + a_{22}) \\ a_1 &= a_{11}a_{22} - a_{23}a_{32} \\ a_2 &= a_{11}a_{23}a_{32} \\ a_0a_1 - a_2 &= -(a_{11})^2 a_{22} - a_{11}(a_{22})^2 + a_{22}a_{23}a_{32} \end{aligned}$$

According to Routh-Hurwitz criteria, the point E_{23} is locally asymptotically stable for

$$\bar{y} > \frac{A_1}{w_1} \tag{5.5}$$

7) The coefficients of jacobian matrix at E^* are

$$a_{11} = -x^* + \frac{w_1x^*y^*}{(x^*+A_1)^2} - \frac{\alpha_1x^*y^*z^*}{(x^*+y^*)^2},\ a_{12} = -\frac{w_1x^*}{(x^*+A_1)} + \frac{\alpha_1x^{*2}z^*}{(x^*+y^*)^2},\ a_{13} = -\frac{\alpha_1x^{*2}}{(x^*+y^*)} < 0,$$
$$a_{21} = \frac{w_2y^{*2}}{(x^*+A_2)^2} + \frac{\alpha_2y^{*2}z^*}{(x^*+y^*)^2} > 0,\ a_{22} = -\frac{w_2y^*}{(x^*+A_2)} - \frac{\alpha_2x^*y^*z^*}{(x^*+y^*)^2} < 0,\ a_{23} = -\frac{\alpha_2y^{*2}}{(x^*+y^*)}$$
$$a_{31} = \frac{\beta_1x^*z^*(x^*+2y^*) - \beta_2z^*y^{*2}}{(x^*+y^*)^2},\ a_{32} = \frac{-\beta_1x^{*2}z^* + \beta_2y^*z^*(2x^*+y^*)}{(x^*+y^*)^2},\ a_{33} = 0$$

Here

$$\begin{aligned} a_0 &= -(a_{11} + a_{22}) \\ a_1 &= a_{11}a_{22} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32} \\ a_2 &= a_{11}a_{23}a_{32} - a_{12}a_{23}a_{31} - a_{21}a_{13}a_{32} + a_{13}a_{31}a_{22} \\ a_0a_1 - a_2 &= -(a_{11}+a_{22})(a_{11}a_{22} - a_{12}a_{21}) + a_{11}a_{13}a_{31} + a_{12}a_{23}a_{31} + a_{22}a_{23}a_{32} + a_{32}a_{21}a_{13} \end{aligned}$$

The point E^* is locally asymptotically stable for $a_0 > 0$, $a_2 > 0$ and $a_0a_1 - a_2 > 0$.

To show the existence of transcritical bifurcation about two axial points, Sotomayor's theorem is applied. The results are stated below:

Theorem 2. The food-web system (2.2) admits the transcritical bifurcation about $E_1 = (1,0,0)$ at $w_3 = w_3^* = \beta_1$.

Proof. The Jacobian matrix about E_1 at $w_3 = \beta_1$ is computed as

$$J_{E_1} = \begin{bmatrix} -1 & -\frac{w_1}{1+A_1} & -\alpha_1 \\ 0 & r & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let the eigenvectors of J_{E_1} and $J_{E_1}^T$ corresponding to zero eigenvalue be V=$(-\alpha_1, 0, 1)^T$ and W=$(0,0,1)^T$ respectively. Denoting F=$[F_1\ F_2\ F_3]^T$. Applying the Sotomayor's theorem, the following conditions are evaluated:

- $W^T F_{w_3}(E_1, w_3^*)$=0

- $W^T[DF_{w_3}(E_1,w_3^*)V] = -1 < 0$, where

$$DF_{w_3}(E_1,w_3^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Let us denote $D^2F(V,V)$ as $\Sigma_{i=1}^{3}\Sigma_{j=1}^{3}\frac{\partial^2 F}{\partial x_i \partial x_j}v_i v_j$ then $W^T[D^2F(E_1,w_3^*)(V,V)] = -2\alpha_1\beta_1 \neq 0\ \forall\ \alpha_1,\ \beta_1$.

All the conditions of Sotomayor's theorem are satisfied, hence the food-web system (2.2) undergoes the transcritical bifurcation about E_1.

Theorem 3. The food-web system (2.2) admits the transcritical bifurcation about $E_2 = (0, \frac{rA_2}{w_2}, 0)$ at $w_3 = w_3^* = \frac{rA_2\beta_2}{w_2}$.

Proof. The Jacobian matrix about E_2 at $w_3 = \frac{rA_2\beta_2}{w_2}$ is evaluated as

$$J_{E_2} = \begin{bmatrix} 1 - \frac{rA_2 w_1}{A_1 w_2} & 0 & 0 \\ r^2 w_2 & -r & -\frac{r\alpha_2 A_2}{w_2} \\ 0 & 0 & 0 \end{bmatrix}$$

Let the eigenvectors of J_{E_2} and $J_{E_2}^T$ corresponding to zero eigenvalue be V=$(0, \alpha_2 A_2, w_2)^T$ and W=$(0,0,1)^T$ respectively. Denoting F=$[F_1\ \ F_2\ \ F_3]^T$. The conditions of Sotomayor's theorem are computed as

- $W^T F_{w_3}(E_2,w_3^*)$=0

- $W^T[DF_{w_3}(E_2,w_3^*)V] = -1 < 0$, where

$$DF_{w_3}(E_2,w_3^*) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

- Let us denote $D^2F(V,V)$ as $\Sigma_{i=1}^{3}\Sigma_{j=1}^{3}\frac{\partial^2 F}{\partial x_i \partial x_j}v_i v_j$ then $W^T[D^2F(E_2,w_3^*)(V,V)] = \alpha_2\beta_2 A_2 w_2 \neq 0\ \forall\ \alpha_2,\ \beta_2,\ A_2,\ w_2$.

Hence the food-web system (2.2) undergoes the transcritical bifurcation about E_2.

5.2 Global stability of boundary points

In this subsection, global stability of boundary equilibrium points E_{12}, E_{13} and E_{23} are established under certain parametric restrictions.

Theorem 4. The equilibrium point $E_{12} = (\hat{x}, \hat{y}, 0)$ is globally asymptotically stable in the interior of the positive quadrant of $x-y$ plane, provided

$$w_1\hat{y}\left(\frac{1}{A_2} + \frac{1}{A_1}\right) < (\hat{x} + A_1)\left(1 - \frac{w_1}{A_1}\right) \text{ and } (A_2+1)\{A_2(\hat{x}+A_1) + A_1\hat{y}\} < A_1A_2(\hat{x}+A_2) \tag{5.6}$$

Proof. Consider a liapunov function

$$L_1 = k_1\left(x - \hat{x} - \hat{x}\,ln\,\frac{x}{\hat{x}}\right) + k_2\left(y - \hat{y} - \hat{y}\,ln\,\frac{y}{\hat{y}}\right)$$

Compute first derivative

$$\frac{dL_1}{dt} = k_1(x-\hat{x})\left(1 - x - \frac{w_1 y}{x + A_1}\right) + k_2(y - \hat{y})\left(r - \frac{w_2 y}{x + A_2}\right)$$

At equilibrium point we have,

$$1 = \hat{x} + \frac{w_1\hat{y}}{\hat{x} + A_1};\ \ r = \frac{w_2\hat{y}}{\hat{x} + A_2}$$

$$\frac{dL_1}{dt} = k_1(x-\hat{x})\left[-(x-\hat{x}) + \frac{w_1\hat{y}}{\hat{x}+A_1} - \frac{w_1 y}{x+A_1}\right] + k_2(y-\hat{y})\left[\frac{w_2\hat{y}}{\hat{x}+A_2} - \frac{w_2 y}{x+A_2}\right]$$

$$\frac{dL_1}{dt} = k_1(x-\hat{x})\left[-(x-\hat{x}) + \frac{w_1\hat{y}(x-\hat{x}) - w_1\hat{x}(y-\hat{y}) - w_1A_1(y-\hat{y})}{(x+A_1)(\hat{x}+A_1)}\right] + k_2\frac{w_2(y-\hat{y})}{(x+A_2)(\hat{x}+A_2)}[-A_2(y-\hat{y}) + \hat{y}(x-\hat{x}) - \hat{x}(y-\hat{y})]$$

Choose, $k_1 = (\hat{x}+A_1),\ k_2 = \dfrac{w_1(\hat{x}+A_2)}{w_2}$

$$\frac{dL_1}{dt} = (x-\hat{x})^2\left[-(\hat{x}+A_1) + \frac{w_1\hat{y}}{x+A_1}\right] - w_1\frac{(\hat{x}+A_2)}{x+A_2}(y-\hat{y})^2 + (x-\hat{x})(y-\hat{y})\left[\frac{w_1\hat{y}}{x+A_2} - \frac{w_1(\hat{x}+A_1)}{x+A_1}\right]$$

$$\frac{dL_1}{dt} \le (x-\hat{x})^2\left[-(\hat{x}+A_1) + \frac{w_1\hat{y}}{x+A_1}\right] - w_1\frac{(\hat{x}+A_2)}{x+A_2}(y-\hat{y})^2 + \frac{(x-\hat{x})^2+(y-\bar{y})^2}{2}\left[\frac{w_1\hat{y}}{x+A_2} + \frac{w_1(\hat{x}+A_1)}{x+A_1}\right]$$

$$\frac{dL_1}{dt} \le \left[-(\hat{x}+A_1) + \frac{w_1\hat{y}}{A_1} + \frac{w_1\hat{y}}{A_2} + \frac{w_1(\hat{x}+A_1)}{A_1}\right](x-\hat{x})^2 + (y-\bar{y})^2\left[-\frac{w_1(\hat{x}+A_2)}{x+A_2} + \frac{w_1(\hat{x}+A_1)}{A_1} + \frac{w_1\hat{y}}{A_2}\right]$$

$$\frac{dL_1}{dt} = \left[-(\hat{x}+A_1) + \frac{w_1\hat{y}}{A_1} + \frac{w_1\hat{y}}{A_2} + \frac{w_1(\hat{x}+A_1)}{A_1}\right](x-\hat{x})^2 + \frac{w_1(\hat{x}+A_1)}{A_1(x+A_2)}(y-\bar{y})^2\left[(x+A_2) - \frac{A_1(\hat{x}+A_2)}{(\hat{x}+A_1)} + \frac{A_1\hat{y}(x+A_2)}{A_2(\hat{x}+A_1)}\right]$$

From (3.1), $\ni \mu > 1$ such that $\left[\mu + A_2 - \dfrac{A_1(\hat{x}+A_2)}{(\hat{x}+A_1)} + \dfrac{A_1\hat{y}(\mu+A_2)}{A_2(\hat{x}+A_1)}\right] < 0$

$\dfrac{dL_1}{dt}$ is negative definite if the condition (5.6) holds. Hence E_{12} is globally stable.

Theorem 5. The equilibrium point $E_{13} = (\tilde{x}, 0, \tilde{z})$ is globally asymptotically stable in the interior of the positive quadrant of $x-z$ plane.

Proof. Consider a liapunov function

$$L_2 = \left(x - \tilde{x} - \tilde{x}\, ln\, \frac{x}{\tilde{x}}\right) + k_1\left(z - \tilde{z} - \tilde{z}\, ln\, \frac{z}{\tilde{z}}\right)$$

Compute the first derivative

$$\begin{aligned}\frac{dL_2}{dt} &= \frac{(x-\tilde{x})}{x}\frac{dx}{dt} + k_1\frac{(z-\tilde{z})}{z}\frac{dz}{dt}\\ &= (x-\tilde{x})(1-x-\alpha_1 z) + k_1(z-\tilde{z})(-w_3+\beta_1 x)\\ &= -(x-\tilde{x})^2 + (x-\tilde{x})(z-\tilde{z})(k_1\beta_1 - \alpha_1)\end{aligned}$$

For $k_1 = \dfrac{\alpha_1}{\beta_1},\ \dfrac{dL_2}{dt} < 0$

Since $\dfrac{dL_2}{dt}$ is negative definite, the point E_{13} is globally asymptotically stable.

Theorem 6. The equilibrium point $E_{23} = (0, \bar{y}, \bar{z})$ is globally asymptotically stable in the interior of the positive quadrant of $y-z$ plane.

Proof. Consider a liapunov function

$$L_3 = \left(y - \bar{y} - \bar{y}\, ln\, \frac{y}{\bar{y}}\right) + k_1\left(z - \bar{z} - \bar{z}\, ln\, \frac{z}{\bar{z}}\right)$$

Compute first derivative

$$\begin{aligned}\frac{dL_3}{dt} &= \frac{(y-\bar{y})}{y}\frac{dy}{dt} + k_1\frac{(z-\bar{z})}{z}\frac{dz}{dt}\\ &= (y-\bar{y})\left(r - \frac{w_2 y}{A_2} - \alpha_2 z\right) + k_1(z-\bar{z})(-w_3+\beta_2 y)\\ &= -\frac{w_2}{A_2}(y-\bar{y})^2 + (y-\bar{y})(z-\bar{z})(k_1\beta_2 - \alpha_2)\end{aligned}$$

For $k_1 = \dfrac{\alpha_2}{\beta_2},\ \dfrac{dL_1}{dt} < 0$

Since $\dfrac{dL_3}{dt}$ is negative definite, therefore E_{23} is globally asymptotically stable in the interior of $y-z$ plane.

6 Persistence of the system

Theorem 7. In addition to assumptions (4.2), (4.4) and (4.6) hold, let the hypotheses of the theorems (4), (5) and (6) hold. If

$$w_3 < \frac{A_1\beta_2}{w_1} \tag{6.1a}$$

$$w_3 < \frac{\beta_1\hat{x}^2+\beta_2\hat{y}^2}{\hat{x}+\hat{y}} \tag{6.1b}$$

$$\text{or } w_3 < min\left(\frac{A_1\beta_2}{w_1}, \frac{\beta_1\hat{x}^2+\beta_2\hat{y}^2}{\hat{x}+\hat{y}}\right)$$

Then the system (2.2) persists.

Proof. Here average Liapunov method is used to prove the persistence theorem. Let average liapunov function is given as

$$\sigma(X) = x^{s_1}y^{s_2}z^{s_3}$$

where $s_1,\ s_2$ and $s_3\ >\ 0$. Clearly $\sigma(X)$ is nonnegative function, so we have

$$\begin{aligned}\Psi(X) &= \frac{\sigma(\dot{X})}{\sigma(X)} = s_1\frac{\dot{x}}{x}+s_2\frac{\dot{y}}{y}+s_3\frac{\dot{z}}{z}\\ &= s_1\left(1-x-\frac{w_1y}{x+A_1}-\frac{\alpha_1xz}{x+y}\right)+s_2\left(r-\frac{w_2y}{x+A_2}-\frac{\alpha_2yz}{x+y}\right)+s_3\left(-w_3+\frac{\beta_1x^2+\beta_2y^2}{x+y}\right)\end{aligned}$$

Inequalities (4.2), (4.4) and (4.6) ensure the existence of E_{12}, E_{13} and E_{23} and it is clear from the theorem (4), (5) and (6), that there are no periodic orbits in the interior of the positive quadrant of $x-y$, $y-z$ and $x-z$ planes. The system (2.2) will persist, if the function $\Psi(X)$ will be positive. For a suitable choice of positive constants s_1, s_2 and s_3, the following conditions must be satisfied for persistence:

$$\Psi(E_0) = \Psi(0,0,0) = s_1+s_2r-s_3w_3 > 0 \tag{6.2}$$

$$\Psi(E_1) = \Psi(1,0,0) = s_2r+s_3(\beta_1-w_3) > 0 \tag{6.3}$$

$$\Psi(E_2) = \Psi\left(0,\frac{rA_2}{w_2},0\right) = s_1\left(1-\frac{rw_1A_2}{A_1w_2}\right)+s_3\beta_2\left(\frac{rA_2}{w_2}-\frac{w_3}{\beta_2}\right) > 0 \tag{6.4}$$

$$\Psi(E_{12}) = \Psi(\hat{x},\hat{y},0) = s_3\left(-w_3+\frac{\beta_1\hat{x}^2+\beta_2\bar{y}^2}{x+y}\right) > 0 \tag{6.5}$$

$$\Psi(E_{13}) = \Psi(\tilde{x},0,\tilde{z}) = (s_2r) > 0 \tag{6.6}$$

$$\Psi(E_{23}) = \Psi(0,\bar{y},\bar{z}) = s_1\left(1-\frac{w_1}{A_1}\bar{y}\right) > 0 \tag{6.7}$$

Keeping s_3 fixed at small positive values and choosing s_1 and $s_2 > 0$, $\Psi(E_0)$ can be made positive. Thus the condition (6.2) holds. The function $\Psi(X)$ will be positive at E_{12}, E_{13} and E_{23} under (4.2), (4.4) and (4.6), provided s_1, s_2 and $s_3 > 0$. The conditions (6.5) and (6.7) hold for the inequalities (6.1a) and (6.1b). Hence the system persists under (6.1a) and (6.1b).

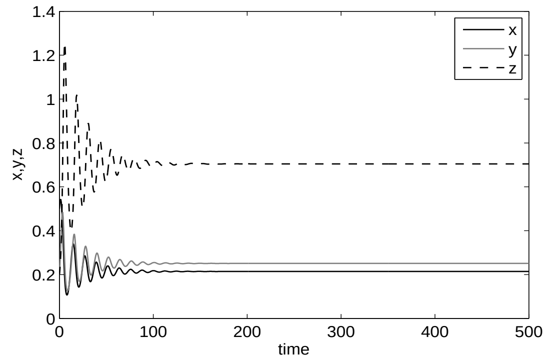
7 Numerical Simulation

In this section, some numerical examples are given to discuss stability, persistence and existence of hopf bifurcation. Consider the following data set:

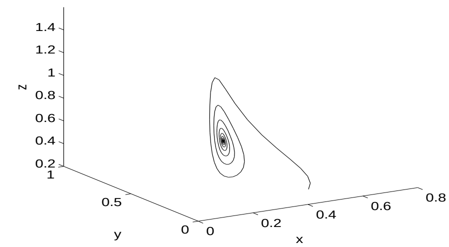
$$\begin{aligned}w_1 &= 0.405, w_2 = 0.3001, A_1 = 1.2, A_2 = 1.5, r = 0.5, \alpha_1 = 2.2, \alpha_2 = 1.2, \beta_1 = 1.5\\ \beta_2 &= 2.6 \text{ and } w_3 = 0.5\end{aligned} \tag{7.1}$$

The interior equilibrium point $E^* \cong (0.2172, 0.2558, 0.7124)$ is locally asymptotically stable and The Fig. 1 verifies the stability of E^*.

Further, all the persistence conditions of theorem (7) are satisfied and the system (2.2) persists at the stable non-trivial equilibrium point.



(a)



(b)

Fig. 1 (a) Time series showing stability and (b) Phase plane diagram stable behaviour of the equilibrium E^* for the set (7.1)

Now consider the following data set:

$$w_1 = 0.405,\ w_2 = 0.3001, w_3 = 0.2,\ A_1 = 1.2, A_2 = 1.5, r = 0.5, \alpha_1 = 2.2, \alpha_2 = 0.4, \beta_1 = 1.5 \text{ and } \beta_2 = 2.6 \tag{7.2}$$

For the above data set, the interior point $E^* \cong (0.304908, 0.301958, 0.272973)$ exists and it is found to be unstable. The Hopf bifurcation occurs at $r \cong 0.305856$ with first lyapunov coefficient as $-3.138546e-002$. The eigenvalues are obtained as $-0.757632,\ \pm i(0.270656)$. Further, the stable limit cycle exists as shown in Fig. 2. All the conditions of persistence as given in theorem (7) are satisfied, which guarantees the coexistence of all three species in the form of periodic solution.

Using MatCont software, another Hopf point is obtained for $r \cong 0.531268$ with first lyapunov coefficient as $9.277797e-004$. When this Hopf point continues, the generalized-Hopf bifurcation occurs at $(0.51335927, 0.44947158)$ in (r, α_2) plane with second lyapunov coefficient $-3.308800e-003$.

The Bogdanov-Takens bifurcation point is detected in (r, α_2) plane at $(0.02014078, 0.00073795)$. At these values, the point will be $E^* \cong (0.053505, 0.101258, 1.201413)$ and corresponding eigenvalues are $-0.669736,\ \pm i(0.00062)$. The normal form coefficients are $(a, b) = (-7.657921e-004, -7.220110e-002)$. The codimension-2 bifurcation diagram is drawn in Fig. 3.

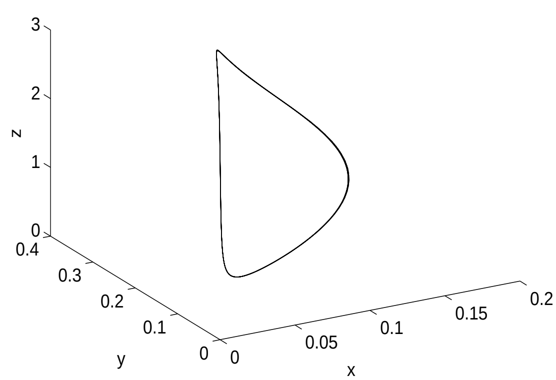


Fig. 2 Phase portrait showing the stable limit cycle for $r \in (0.31, 0.53)$.

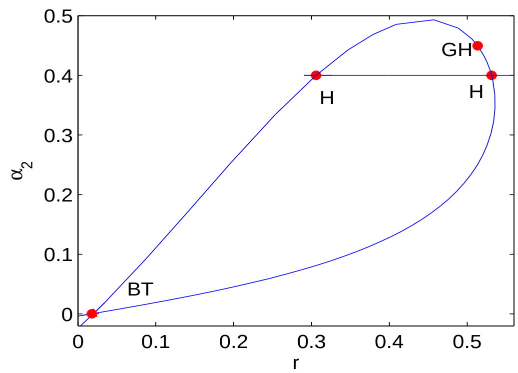


Fig. 3 Two parameter bifurcation diagram in (r, α_2) plane, showing the existence of Hopf, generalized-Hopf and Bogdanov-Takens bifurcation for the initial condition $(0.2, 0.2, 0.2)$

8 Discussion

In this paper, a three species modified leslie- gower dynamical system with switching predator is studied. Dynamical behaviour of all possible equilibrium points is discussed and it is found that three equilibrium points are unconditionally unstable. Global stability of all the planar points have been investigated in the positive octant of their respective planes. The existence of transcritical bifurcation has been shown about two axial points, when mortality rate is chosen as bifurcation parameter. This analysis has not been done in earlier work on switching. It has been shown that survival of all three species is possible in form of stability as well as periodic solution. The existence of Hopf bifurcation and codimension-2 bifurcations such as generalized-Hopf and Bogdanov-Takens bifurcations have been investigated through numerical simulation about the positive interior point, when the parameters involved in the dynamics of Leslie-Gower predator are chosen as bifurcation parameters.

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